TITLE: Conjugate Silhouette Nets

Conjugate Silhouette Nets

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Abstract. Conjugate nets, Laplace transformations and projective translation surfaces are exploited for CAGD purposes. The latter are shown to be equivalent with conjugate nets having degenerated Laplace transforms. Relations to conjugate nets with planar silhouettes, supercyclides and Dupin cyclides are given.

§1. Conjugate Nets and their Laplace Transforms

Conjugate nets play an important role in classical differential geometry, especially because of their projective invariance (see [3]). Representing a surface in d-space by *homogeneous* coordinates

$$\mathcal{S} \quad \ldots \quad \boldsymbol{x} : D \to \mathbb{R}^{d+1},\ D \subset \mathbb{R}^2,\ \ \boldsymbol{x} \in C^\infty[D], \tag{1}$$

where D is an open connected domain of the "parameter plane" $\mathbb{R}^2$, then a conjugate net is defined by the validity of a Laplacian equation

$$\boldsymbol{x}_{uv} + a\boldsymbol{x}_u + b\boldsymbol{x}_v + c\boldsymbol{x} = 0 \tag{2}$$

with certain functions $a, b, c \in C^\infty[D]$. To exclude planar surfaces

$$\dim \operatorname{span}(\boldsymbol{x}, \boldsymbol{x}_u, \boldsymbol{x}_v, \boldsymbol{x}_{uu}, \boldsymbol{x}_{vv}) \geq 3, \quad d \geq 3 \tag{3}$$

is assumed throughout. The geometric meaning of (2) for $d = 3$ is that the characteristic lines of the tangent planes along one isoparametric line are tangent to the other. A second characterization is that the two tangents to the isoparametric lines have a *harmonic cross ratio* with the two asymptotic tangents.

The most important property of conjugate nets is that they have in both isoparametric directions a *Laplace transformed net* (*Laplace transform*) which is defined by the striction points of the two developables generated by the

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tangent planes along the net curves (isoparametric lines). Analytically, they are given by

$$\mathcal{L}_u(\boldsymbol{x}) \quad \ldots \quad \boldsymbol{y} := \boldsymbol{x}_u + b\boldsymbol{x}, \tag{4}$$

$$\mathcal{L}_v(\boldsymbol{x}) \quad \ldots \quad \boldsymbol{z} := \boldsymbol{x}_v + a\boldsymbol{x}. \tag{5}$$

Indeed, by (2), we get

$$\boldsymbol{y}_v = h\boldsymbol{x} - a\boldsymbol{y}, \tag{6}$$

$$\boldsymbol{z}_u = k\boldsymbol{x} - b\boldsymbol{z} \tag{7}$$

with

$$h = b_v + ab - c, \quad k = a_u + ab - c \tag{8}$$

being the so-called Darboux invariants. Eqn. (6) shows that the line $\boldsymbol{x} \wedge \boldsymbol{y}$ is tangent to the u-curve (isoparametric line with v = const, u varying) on the surface $(\boldsymbol{x})_D$ as well as tangent to the v-curve on $(\boldsymbol{y})_D$; analogously $\boldsymbol{x} \wedge \boldsymbol{z}$ is tangent to the v-curve on $(\boldsymbol{x})_D$ and the u-curve on $(\boldsymbol{z})_D$.

In euclidean differential geometry, conjugate nets owe their importance to the fact that the *curvature lines* on every surface are conjugate (except at umbilic points). Thus, for CAGD applications, one can profit from the theory of conjugate nets since, for instance, the two families of circles on a Dupin cyclide or the net of profile curves and meridian circles on a surface of revolution is a conjugate net.

In this paper we deal with the special case that the two Laplace transforms *degenerate* into curves. Assuming that the net curves have nowhere asymptotic directions, i.e.

$$\boldsymbol{x} \wedge \boldsymbol{x}_u \wedge \boldsymbol{x}_{uu} \neq 0\,, \quad \boldsymbol{x} \wedge \boldsymbol{x}_v \wedge \boldsymbol{x}_{vv} \neq 0, \tag{9}$$

then the degeneration conditions $\boldsymbol{y} \wedge \boldsymbol{y}_u \wedge \boldsymbol{y}_v = 0$ and $\boldsymbol{z} \wedge \boldsymbol{z}_u \wedge \boldsymbol{z}_v = 0$ imply

$$h = 0, \; k = 0 \quad \text{for all } (u,v) \in D. \tag{10}$$

Definition 1. *A conjugate net in 3-space satisfying (10) with regularity conditions (3), (9) is called* a conjugate silhouette net.

This notation is justified since by (4), (6), (10) all u-tangents along a v-curve meet at the *fixed* point $\boldsymbol{y}$, thus building up a general cone with apex $\boldsymbol{y}$. Thus the v-curve $\boldsymbol{x}(u_0, v)$ can be considered as a silhouette on the surface $\mathcal{S}$ by central illumination from $\boldsymbol{y}$. Similarily the u-curves are silhouette lines by central illumination from $\boldsymbol{z}$.

On the other hand, it is easy to see that a net of silhouette lines on a surface whereby the centers of illumination vary on two curves is automatically conjugate, provided that the generators of the enveloping cones are tangent to

the net curves. Thus, the equations (2), (10) characterize conjugate silhouette nets (up to degenerated cases).

At first glance, this class of surfaces seems to be very restricted. But this is not true: it comprises many subclasses of surfaces considered in CAGD literature such as Dupin cyclides ([2,4,7]), supercyclides ([1,5,8,9]), nets with planar silhouettes ([6]) etc.. As for all of these examples, the whole class of conjugate silhouette nets is well-suited for CAGD applications, in particular for geometric modelling purposes because of their simple blending properties: Putting two of them together along a common net curve immediately yields a G^1-continuity, once the corresponding centers of illumination coincide.

But there is still another reason making these surfaces worth considering in CAGD: They admit a very simple generation as so-called "projective translation surfaces", as will be derived in the next section.

§2. Projective Translation Surfaces

Let

$$\mathcal{C}_1 \ \ldots \ \boldsymbol{p} : I_1 \to \mathbb{R}^{d+1}, \ \ \mathcal{C}_2 \ \ldots \ \boldsymbol{q} : I_2 \to \mathbb{R}^{d+1} \tag{11}$$

be two C^∞-curves in d-space represented also in homogeneous coordinates (I_1, I_2 being two open nonvoid intervals of $\mathbb{R}$). Then one gets a surface $\mathcal{S}$ (1) simply by setting

$$\mathcal{S} \ \ldots \ \boldsymbol{x}(u,v) := \boldsymbol{p}(u) + \boldsymbol{q}(v), \quad (u,v) \in I_1 \times I_2 := D \tag{12}$$

Definition 2. *Surfaces defined by (12) via two curves (11) are called* projective translation surfaces.

This definition generalizes the usual euclidean (or affine) definition of translation surfaces, where the same formula (12) is used but interpreted in affine (non-homogeneous) coordinates. So one curve can be considered to move along the other thus sweeping out the surface. In the projective case, the generating point $\boldsymbol{x}(u,v)$ always lies on the line $\boldsymbol{p}(u) \wedge \boldsymbol{q}(v)$ joining these two points of $\mathcal{C}_1$ and $\mathcal{C}_2$ independently. It must be noticed that the *normalizations* are essential (not arbitrarily to be choosen like usually when dealing with curves): they determine the *position* of that point $\boldsymbol{x}(u,v)$ on the line $\boldsymbol{p}(u) \wedge \boldsymbol{q}(v)$.

Now we can establish one of our main results:

Theorem 1. *Every conjugate silhouette net is a projective translation surface, and the net curves correspond to the isoparameter lines in the representation (12).*

Proof: We have, by definition, $h = 0, \ \ k = 0$, and hence in particular

$$a_u = b_v. \tag{13}$$

Assuming D to be simply connected, we conclude that there exists a C^∞-function $f : D \to \mathbb{R}$ with

$$f_u = b, \quad f_v = a. \tag{14}$$

Taking $\rho := e^f$, we calculate $\rho_u = \rho b$, $\rho_v = \rho a$ and $\rho_{uv} = (\rho b)_v = \rho(ab + b_v) = \rho c$, the latter observing (8), (10). Since $\rho \neq 0$ in D, we can *renormalize* $\overline{\boldsymbol{x}} := \rho \boldsymbol{x}$, obtaining

$$\overline{\boldsymbol{x}}_{uv} = (\rho_{uv} - \rho c)\boldsymbol{x} + (\rho_u - \rho b)\boldsymbol{x}_v + (\rho_v - \rho a)\boldsymbol{x}_u, \tag{15}$$

and thus

$$\overline{\boldsymbol{x}}_{uv} = 0. \tag{16}$$

This equation immediately yields a representation (12) by integration (possibly restricted to a rectangle $I_1 \times I_2$ within D). $\square$

Up to now we think of that renormalization as always having been done, so the Laplace equation (2) has the coefficients

$$a = 0, \ b = 0, \ c = 0. \tag{17}$$

Therefore, the Laplace transforms (4) and (5) are now given by

$$\mathcal{L}_u(\boldsymbol{x}) \ \ldots \ \boldsymbol{y}(u) = \frac{d\boldsymbol{p}(u)}{du}, \qquad u \in I_1, \tag{18}$$

$$\mathcal{L}_u(\boldsymbol{x}) \ \ldots \ \boldsymbol{z}(v) = \frac{d\boldsymbol{q}(v)}{dv}, \qquad v \in I_2. \tag{19}$$

Calling these curves the projective hodographs of $\boldsymbol{p}$ and $\boldsymbol{q}$ respectively, we can state

Corollary. *The Laplace transforms of a conjugate silhouette net $\mathcal{C}$ are the projective hodographs of the generating curves C_1, C_2 of $\mathcal{C}$ (considered as a projective translation surface).*

§3. Axial Silhouette Nets

Definition 3. *A conjugate silhouette net is called* axial *if the generating curves C_1, C_2 in its representation (12) as a projective translation surface are (parts of) straight lines. These lines are called the* first *and the* second axis *of the net.*

The conditions for axial conjugate silhouette nets are that $\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{p}''$ and likewise $\boldsymbol{q}$, $\boldsymbol{q}'$, $\boldsymbol{q}''$ must be linearly dependent (a prime at $\boldsymbol{p}$ indicating derivation with respect to u and at $\boldsymbol{q}$ with respect to v). Assuming $\boldsymbol{p}$, $\boldsymbol{p}'$ and likewise $\boldsymbol{q}$, $\boldsymbol{q}'$ to be linearly independent (otherwise the point would be stationary) we have

$$\boldsymbol{p}'' = \alpha \boldsymbol{p} + \beta \boldsymbol{p}', \quad \boldsymbol{q}'' = \gamma \boldsymbol{q} + \delta \boldsymbol{q}' \tag{20}$$

with some C^∞-functions α, β of u and γ, δ of v characterizing axial nets.

This has many consequences; most of them we proved earlier for supercyclides and for nets with planar silhouettes [5,6]. Now we give the result for the general case of axial silhouette nets:

Theorem 2.

a) *All the net curves (of both families) are planar curves,*

b) *The planes of the net curves of each family belong to a pencil,*

c) *The axes of these two pencils coincide with the second and the first axis of the net (i. e. the plane of a curve of the first [second] family passes through the second [first] axis,*

d) *The apexes of the envelopping cones (the "light centers") along a u-curve [v-curve] lie on the second [first] axis,*

e) *Any two u-curves [v-curves] are projectively equivalent to each other.*

Proof: We perform the proofs only for the u-curves; the assertions with respect to the v-curves follow analogously.

a): From (12), (20) we derive

$$\boldsymbol{x}_u = \boldsymbol{p}', \quad \boldsymbol{x}_{uu} = \alpha \boldsymbol{p} + \beta \boldsymbol{p}', \tag{21}$$

$$\boldsymbol{x}_{uuu} = (\alpha' + \alpha\beta)\boldsymbol{p} + (\alpha + \beta' + \beta^2)\boldsymbol{p}'. \tag{22}$$

Hence $\boldsymbol{x}_u \wedge \boldsymbol{x}_{uu} \wedge \boldsymbol{x}_{uuu} = 0$, meaning that the u-curves are planar.

b), c): Eqns. (21) show that $\boldsymbol{p}$, $\boldsymbol{p}'$ are contained in the plane $\boldsymbol{x} \wedge \boldsymbol{x}_u \wedge \boldsymbol{x}_{uu}$ of this u-curve; but $\boldsymbol{p}$, $\boldsymbol{p}'$ span the first axis.

d): The apex of the envelopping cone is given by (21) as $\boldsymbol{p}'$. Hence it is lying on the first axis.

e): Assumption (9) implies $\alpha \neq 0$ for all $u \in I_1$. Thus, $\boldsymbol{x}_u$ and $\boldsymbol{x}_{uu}$ can be eliminated from (21) and with this Eqn. (22) yields

$$\boldsymbol{x}_{uuu} = \left(\frac{\alpha'}{\alpha} + \beta\right) \boldsymbol{x}_{uu} + \left(\frac{\alpha'}{\alpha}\beta + \alpha + \beta'\right) \boldsymbol{x}_u. \tag{23}$$

Thus the coefficient of the fundamental equation (see [3]) do not depend on the second parameter v; this means geometrically that all u-curves are projectively equivalent. □

For CAGD purposes *rational* (and polynomial) surfaces are of particular interest. The explicit representation (12) makes it very easy to pick out rational surfaces from that general class: The only thing one has to do is to insert *rational representations* for $\boldsymbol{p}(u)$ and $\boldsymbol{q}(v)$. We restrict this procedure to axial nets, and derive from it the (rational) Bézier representation.

Theorem 3. *For any pair of planar rational curves C_1 and C_2, there exists an axial conjugate silhouette net having its u-curves projectively equivalent to C_1 and its v-curves projectively equivalent to C_2. The axes can be arbitrarily prescribed as two skew straight lines.*

Proof: Let C_1, C_2 be represented in planar homogeneous coordinates by triples of linearly independent polynomials

$$C_1 \ \ldots \ x_i = f_i(u) \ (i = 0,1,2), \quad C_2 \ \ldots \ x_i = g_i(v) \ (i = 0,1,2). \tag{24}$$

Furthermore, let the axes be spanned by vectors $\boldsymbol{a}$, $\boldsymbol{b} \in \mathbb{R}^4$ and $\boldsymbol{c}$, $\boldsymbol{d} \in \mathbb{R}^4$ resp. Then $det(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) \neq 0$ since the axes are assumed to be skew. With this we can set

$$\boldsymbol{p}(u) := \frac{1}{f_0(u)}\big(f_1(u)\boldsymbol{a} + f_2(u)\boldsymbol{b}\big), \quad \boldsymbol{q}(v) := \frac{1}{g_0(v)}\big(g_1(v)\boldsymbol{c} + g_2(v)\boldsymbol{d}\big) \quad (25)$$

getting the desired axial net $\mathcal{S}$ by (12) (restricted to intervals $I_1, I_2 \subset \mathbb{R}$ where $f_0(u)$ resp. $g_0(v)$ have no zeros). Now, indeed, the u-curves are planar and projectively equivalent to $\mathcal{C}_1$ since, for fixed $v = v_0$, we have $\overline{\boldsymbol{p}}(u) = f_0(u)\,\boldsymbol{p}(u) = f_0(u)\boldsymbol{q}(v_0) + f_1(u)\boldsymbol{a} + f_2(u)\boldsymbol{b}$ so that $f_i(u)$ are the coordinates with respect to basis $\boldsymbol{q}(v_0)$, $\boldsymbol{a}$, $\boldsymbol{b}$. The proof for the v-curves follows the same line, mutatis mutandis. □

Obviously, the representation (25) is not unique. But we can immediately derive from (12) and (25) the Bézier representation of $\mathcal{S}$: First renormalize (12) with the factor $f_0(u)g_0(v)$ getting

$$\overline{\boldsymbol{x}}(u,v) \;=\; g_0(v)\big(f_1(u)\boldsymbol{a} + f_2(u)\boldsymbol{b}\big) \;+\; f_0(u)\big(g_1(v)\boldsymbol{c} + g_2(v)\boldsymbol{d}\big), \qquad (26)$$

and then we expand the polynomials $f_i(u)$ and $g_i(v)$ with repsect to the Bernstein basis

$$f_i(u) = \sum_{j=0}^{n} \alpha_{i,j} B_j^n(u), \quad g_k(v) = \sum_{l=0}^{m} \beta_{k,l} B_l^m(v), \qquad (27)$$

getting the usual homogeneous Bézier representation

$$\overline{\boldsymbol{x}}(u,v) \;=\; \sum_{j=0}^{n}\sum_{l=0}^{m} \boldsymbol{b}_{j,l} B_j^n(u)\, B_k^m(v) \qquad (28)$$

with the control points

$$\boldsymbol{b}_{j,l} = \beta_{0,l}(\alpha_{1,j}\boldsymbol{a} + \alpha_{2,j}\boldsymbol{b}) \;+\; \alpha_{0,j}(\beta_{1,l}\boldsymbol{c} + \beta_{2,l}\boldsymbol{d}). \qquad (29)$$

Since these calculations can be done also backwards, we obtain

Corollary. *The conditions (29) for the control points characterize a (n,m)-rational Bézier surface (28) to be an axial conjugate silhouette net.*

§4. Applications to Dupin Cyclides

Dupin cyclides are special kinds of supercyclides [5], and therefore they should have representations as axial conjugate silhouette nets so far they have fourth order and skew axes. However, their usual representation starts from the midpoint curves $\boldsymbol{Y}(u)$ and $\boldsymbol{Z}(v)$ of the two families of envelopping spheres and results in

$$\mathcal{D} \;\ldots\; \boldsymbol{X}(u,v) = \frac{r_2(v)\boldsymbol{Y}(u) + r_1(u)\boldsymbol{Z}(v)}{r_2(v) + r_1(u)}, \qquad (30)$$

where r_1, r_2 denote the corresponding radius functions. Though this is also a sum of two vectors it has neither the form (12) nor are the components parts of straight lines.

Thus, the question arises of how to transform (30) into (12). The first step to solve this problem consists in passing to homogeneous coordinates $\boldsymbol{x} = (x_0, x_1, x_2, x_3)^T$

$$\boldsymbol{x} = \rho \begin{pmatrix} 1 \\ X \end{pmatrix}, \quad \rho \in \mathbb{R} \setminus \{0\} \tag{31}$$

(i.e. $x_0 = \rho \cdot 1$, $x_i = \rho \cdot X_i$ $(i = 1,2,3)$) and to take in the present case $\rho = \frac{1}{r_1} + \frac{1}{r_2}$ yielding indeed (12)

$$\boldsymbol{p}(u) = \begin{pmatrix} \dfrac{1}{r_1(u)} \\ \dfrac{1}{r_1(u)} \boldsymbol{Y}(u) \end{pmatrix}, \quad \boldsymbol{q}(v) = \begin{pmatrix} \dfrac{1}{r_2(v)} \\ \dfrac{1}{r_2(u)} \boldsymbol{Z}(u) \end{pmatrix}. \tag{32}$$

However, the curves $\boldsymbol{p}$ and $\boldsymbol{q}$ describe an ellipse and a hyperbola as before. Obviously, another representation of the same kind (12) must satisfy

$$\overline{\boldsymbol{p}}(u) = \boldsymbol{p}(u) + \boldsymbol{c}, \quad \overline{\boldsymbol{q}}(v) = \boldsymbol{q}(v) - \boldsymbol{c} \tag{33}$$

with a *constant* vector $\boldsymbol{c}$.

Starting with the explicit representations

$$\boldsymbol{Y}(u) = \begin{pmatrix} 1 \\ \dfrac{1-u^2}{1+u^2}\rho \\ \dfrac{2u}{1+u^2}\rho\omega \\ 0 \end{pmatrix}, \qquad \boldsymbol{Z}(v) = \begin{pmatrix} 1 \\ \dfrac{1+v^2}{1-v^2}\rho\sigma \\ 0 \\ \dfrac{2v}{1-v^2}\rho\omega \end{pmatrix} \tag{34}$$

(with some shape parameters $d, \rho, \sigma, |\sigma| < 1$, $\omega = \sqrt{1-\sigma^2}$) and observing the corresponding radius functions

$$r_1(u) = \frac{f_0(u)}{1+u^2}, \quad r_2 = \frac{g_0(v)}{1-v^2}, \tag{35}$$

whereby

$$f_0(u) = (1+u^2)d\rho - (1-u^2)\rho\sigma, \; g_0(v) = (1+v^2)\rho - (1-v^2)d\sigma,$$

we finally obtain

$$\overline{\boldsymbol{p}}(u) = \frac{1}{f_0(u)} \left(\left((1+u^2) - \frac{1}{d\sigma} f_0(u)\right) \begin{pmatrix} 1 \\ d \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2\rho\omega u \\ 0 \end{pmatrix} \right) \tag{36}$$

and

$$\overline{q}(u)=\frac{1}{g_0(u)}\left(\left((1-v^2)+\frac{1}{d\sigma}g_0(v)\right)\begin{pmatrix}1\\ d\sigma^2\\ 0\\ 0\end{pmatrix}+\begin{pmatrix}0\\ 0\\ 0\\ 2\rho\omega v\end{pmatrix}\right) \tag{37}$$

with

$$c=\frac{1}{d\sigma}(1,\ 0,\ 0,\ 0)^T.$$

Thus we proved

Theorem 4. *The formulas (36), (37) (inserted into (12)) yield an explicit representation of nonparabolic Dupin cyclides with skew axes as axial conjugate silhouette nets with respect to a suitable (homogeneous) coordinate system.*

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